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THE KECHRIS-WOODIN RANK IS FINER THAN THE ZALCWASSER RANK

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ABSTRACT. For each differentiable function f on the unit circle, the Kechris-Woodin rank measures the failure of continuity of the derivative function f' while the Zalcwasser rank measures how close the Fourier series of f is to being a uniformly convergent series. We show that the Kechris-Woodin rank is finer than the Zalcwasser rank. Roughly speaking, small ranks mean the function is well behaved and big ranks imply bad behavior. For each countable ordinal, we explicitly construct a continuous function with everywhere convergent Fourier series such that the Zalcwasser rank of the function is bigger than the ordinal.

INTRODUCTION

Zalcwasser [Za] introduced a rank that measures the uniform convergence of sequences of continuous functions on the unit interval. We apply the Zalcwasser rank to the Fourier series of a continuous function on the unit circle. Throughout this paper, we will only consider the Zalcwasser rank on the Fourier series of a continuous function. In [AK] it is shown that on EC (the set of all continuous functions, on the unit circle, with convergent Fourier series), the Zalcwasser rank is a $\mathbf{\Pi}^1_1$-norm which is unbounded below ω_1, i.e., functions in EC are arbitrarily bad in terms of this rank. Kechris and Woodin [KW] defined a rank that measures the uniform continuity of the derivative of a differentiable function. We shall refer to this rank as the Kechris-Woodin rank. In fact, they have shown that on the set of all differentiable functions, the Kechris-Woodin rank is a $\mathbf{\Pi}^1_1$-norm which is unbounded below ω_1.

Ajtai and Kechris [AK] conjectured that the Kechris-Woodin rank is finer than the Zalcwasser rank, meaning that for any function f, the Zalcwasser rank is less than or equal to the Kechris–Woodin rank. Also see [Ra]. There is a fair amount of evidence supporting this conjecture. For example, the Zalcwasser rank is 1, i.e., the smallest possible number, for all differentiable functions f, whose derivative f', is bounded. On the other hand, on the set of all differentiable functions with bounded derivatives, the Kechris-Woodin rank is unbounded below ω_1 (see [KW]). Our main result is an affirmative answer to this conjecture of Ajtai and Kechris. We study another problem in [AK]. Namely, can we precisely build for each countable ordinal α a function in EC

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of the Zalcwasser rank exceeding α? With our new definition of the Zalcwasser rank, we are able to provide such a construction. This implies that we have another proof that EC is non-Borel. In fact, Ajtai and Kechris [AK] have shown that EC is $\mathbf{\Pi}_1^1$-complete.

Definitions and background

Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{T}$ denote the unit circle and $C(\mathbb{T})$ the Polish space of continuous functions on $\mathbb{T}$ with the uniform metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{T}\}.$$

$C(\mathbb{T})$ can also be considered as the space of all continuous 2π-periodic functions on $\mathbb{R}$, by viewing $\mathbb{T}$ as $\mathbb{R}/2\pi\mathbb{Z}$. We denote by $D(\mathbb{T})$ the set of differentiable functions on $\mathbb{T}$.

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of positive integers and $\mathbb{N}^{\mathbb{N}}$ the Polish space with the usual product topology, where $\mathbb{N}$ is given the discrete topology. A subset A of a Polish space X is called $\mathbf{\Pi}_1^1$ if there is a Borel function f, from $\mathbb{N}^{\mathbb{N}}$ to X such that the image under f of $\mathbb{N}^{\mathbb{N}}$ is $X - A$, i.e., $f(\mathbb{N}^{\mathbb{N}}) = X - A$. Thus a $\mathbf{\Pi}_1^1$ set is coanalytic. A $\mathbf{\Pi}_1^1$ subset A of X is called complete, if for any Polish space Y and any $\mathbf{\Pi}_1^1$ subset B of Y, there is a Borel function f from Y to X such that the preimage under f of A is B, i.e., $B = f^{-1}(A)$. It is easy to see that all complete $\mathbf{\Pi}_1^1$ sets are non–Borel.

A norm on a set P is any function φ taking P into the ordinals. For each such φ, we associate the prewellordering $\leq_\varphi$ on P, $x \leq_\varphi y \iff \varphi(x) \leq \varphi(y)$. φ is regular if φ maps P onto some ordinal λ. Two norms φ and ψ on P are equivalent if the two associated prewellorderings are the same ($\leq_\varphi = \leq_\psi$), i.e., $\varphi(x) \leq \varphi(y) \iff \psi(x) \leq \psi(y)$. Every norm is equivalent to a unique regular norm. To avoid trivial norms on a set, e.g., the constant function, we make the following restriction. Given a Polish space X and a $\mathbf{\Pi}_1^1$ subset P of X, we say that a norm $\varphi: P \to \text{Ordinals}$ is a $\mathbf{\Pi}_1^1$-norm if there are $\mathbf{\Pi}_1^1$ subsets R and Q of $X \times X$ such that

$$(0) \qquad y \in P \Rightarrow [x \in P \ \&\ \varphi(x) \leq \varphi(y) \iff (x, y) \notin R \iff (x, y) \in Q].$$

It is very well known that if a subset A of a Polish space and its complement are both $\mathbf{\Pi}_1^1$, then A is Borel (see [Mo]). Hence in (0), we see that in a uniform manner for $y \in P$, the set $\{x \in P : \varphi(x) \leq \varphi(y)\}$ is $\mathbf{\Pi}_1^1$ ($(x, y) \in Q$) and the complement of a $\mathbf{\Pi}_1^1$ set ($(x, y) \notin R$), hence a Borel set. In [Mo] it is shown that every $\mathbf{\Pi}_1^1$-norm is equivalent to one which takes values in ω_1, the first uncountable ordinal. One of the basic facts is that every $\mathbf{\Pi}_1^1$ subset P admits a $\mathbf{\Pi}_1^1$-norm $\varphi: P \to \omega_1$ (see [Mo]). Hence it is very natural to look for a canonical norm on $\mathbf{\Pi}_1^1$ sets that arise in analysis, topology, etc. We will introduce $\mathbf{\Pi}_1^1$-norms on the set of continuous functions with everywhere convergent Fourier series and the set of differentiable functions. From rank theory, we have the following fundamental theorem (see Chapter 4, [Mo]).

Boundedness Principle. *Let X be a Polish space. Let P be a $\mathbf{\Pi}_1^1$ subset of X and $\varphi: P \to \omega_1$ be a $\mathbf{\Pi}_1^1$-norm on P. Then P is Borel if and only if φ is bounded below ω_1.*

With this basic principle, one can prove that a $\mathbf{\Pi}_1^1$ set P is $\mathbf{\Pi}_1^1$ non-Borel by showing that some $\mathbf{\Pi}_1^1$-norm on P is unbounded below ω_1.

THE KECHRIS-WOODIN RANK

We define a $\mathbf{\Pi}_1^1$-norm on $D(\mathbb{T})$, which we refer to as the Kechris-Woodin rank. We consider $\mathbb{T}$ as $[0, 2\pi]$ identifying 0 with 2π. When we say U is an open neighborhood in $\mathbb{T}$, U is considered as the usual open set in $\mathbb{R}$. Let f be a function and I an interval with endpoints a and b. We define the following:

$$\Delta f(I) = \frac{f(b) - f(a)}{b - a}.$$

Fix $f \in C(\mathbb{T})$ and $\epsilon > 0$. For each closed subset P of $\mathbb{T}$, we define the K–W derived set of P by

$$\partial_{f,\epsilon}^{KW}(P) = \{x \in P: \forall \text{ open neighborhood } U \text{ of } x,\ \exists \text{ closed intervals } I, J \subseteq U \text{ such that } I \cap J \cap P \neq \varnothing \text{ and } |\Delta f(I) - \Delta f(J)| \geq \epsilon\}.$$

$\partial_{f,\epsilon}^{KW}(P)$ consists of all ϵ badly behaved points of P in terms of the derivative of f. Clearly $\partial_{f,\epsilon}^{KW}(P)$ is closed. We then define the sequence $\langle \partial_{f,\epsilon}^{KW}(P, \alpha)\rangle_{\alpha<\omega_1}$ by transfinite induction. Let

(1) $\partial_{f,\epsilon}^{KW}(P, 0) = P$.
(2) $\partial_{f,\epsilon}^{KW}(P, \alpha+1) = \partial_{f,\epsilon}^{KW}(\partial_{f,\epsilon}^{KW}(P, \alpha))$.
(3) For λ a limit ordinal, $\partial_{f,\epsilon}^{KW}(P, \lambda) = \bigcap_{\alpha<\lambda} \partial_{f,\epsilon}^{KW}(P, \alpha)$.

Note that $\bigcup_{\epsilon>0} \partial_{f,\epsilon}^{KW}(P, \alpha) = \bigcup_{n\in\mathbb{N}} \partial_{f,\frac{1}{n}}^{KW}(P, \alpha)$. We now define the sequence $\langle \partial_f^{KW}(P, \alpha)\rangle_{\alpha<\omega_1}$ by setting

$$\partial_f^{KW}(P, \alpha) = \bigcup_{n\in\mathbb{N}} \partial_{f,\frac{1}{n}}^{KW}(P, \alpha).$$

Fact 1. $f \in D(\mathbb{T}) \iff \exists \alpha < \omega_1,\ \partial_f^{KW}(\mathbb{T}, \alpha) = \varnothing$.

Using this fact, we can define the Kechris–Woodin rank on $D(\mathbb{T})$. For each $f \in D(\mathbb{T})$, let $|f|_{KW} =$ the least ordinal α for which $\partial_f^{KW}(\mathbb{T}, \alpha) = \varnothing$. We let $b_1D(\mathbb{T})$ be the set of all functions whose derivatives are bounded in absolute value by 1. The following two facts appear in [KW].

Fact 2. *For each* $\alpha < \omega_1$, *there is a function* f *in* $b_1D(\mathbb{T})$ *with* $|f|_{KW} = \alpha$.

Fact 3. $|\cdot|_{KW}: D(\mathbb{T}) \to \omega_1$ *is a* $\mathbf{\Pi}_1^1$*-norm.*

By these two facts and the Boundedness Principle, we have the following:

Corollary [KW]. *The sets* $D(\mathbb{T})$ *and* $b_1D(\mathbb{T})$ *are* $\mathbf{\Pi}_1^1$ *non-Borel subsets of* $C(\mathbb{T})$.

THE ZALCWASSER RANK

We associate to each $f \in C(\mathbb{T})$, its Fourier series $S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$, where $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$. Let

$$S_n(f, t) = \sum_{k=-n}^{n} \hat{f}(n)e^{ikt}$$

be the nth partial sum of the Fourier series of f. We say "the Fourier series of f converges at a point $t \in \mathbb{T}$" if the sequence $\langle S_n(f, t)\rangle_{n\in\mathbb{N}}$ converges. We will give a rank on EC, the collection of all continuous functions with everywhere convergent Fourier series. According to a standard theorem [Ka], if the Fourier series of f at t converges, then it must converge to $f(t)$. Hence,

$$\begin{aligned} EC =&\{f \in C(\mathbb{T}) : \forall\, t \in [0, 2\pi],\ \langle S_n(f, t)\rangle_{n\in\mathbb{N}} \text{ converges }\} \\ =&\{f \in C(\mathbb{T}) : \forall\, t \in [0, 2\pi],\ f(t) = \lim_{n\to\infty} S_n(f, t)\}. \end{aligned}$$

Let $f \in C(\mathbb{T})$, $P \subseteq \mathbb{T}$ be a closed set, and let $x \in P$. We define the value of the oscillation function of f on P at x as follows.

$$\omega(x, f, P) = \inf_{\delta>0}\inf_{p\in\mathbb{N}} \sup\{|S_m(f, y) - S_n(f, y)| : \\ m, n \geq p \;\&\; y \in P \;\&\; |x - y| < \delta\}.$$

Thus the oscillation function of f on P measures how bad the uniform convergence of the Fourier series of f, near x, is on P. For each $f \in C(\mathbb{T})$ and each $\epsilon > 0$, define the Z derived set of P by

$$\partial^Z_{f,\epsilon}(P) = \{x \in P : \omega(x, f, P) \geq \epsilon\}.$$

Fix $f \in C(\mathbb{T})$ and $\epsilon > 0$. We define $\langle \partial^Z_{f,\epsilon}(P, \alpha)\rangle_{\alpha<\omega_1}$ by transfinite induction as follows. Let

(1) $\partial^Z_{f,\epsilon}(P, 0) = P$.
(2) $\partial^Z_{f,\epsilon}(P, \alpha + 1) = \partial^Z_{f,\epsilon}(\partial^Z_{f,\epsilon}(P, \alpha))$.
(3) For limit ordinals λ, $\partial^Z_{f,\epsilon}(P, \lambda) = \bigcap_{\alpha<\lambda} \partial^Z_{f,\epsilon}(P, \alpha)$.

Note that $\bigcup_{\epsilon>0} \partial^Z_{f,\epsilon}(P, \alpha) = \bigcup_{n\in\mathbb{N}} \partial^Z_{f,\frac{1}{n}}(P, \alpha)$. We now define the sequence $\langle \partial^Z_f(P, \alpha)\rangle_{\alpha<\omega_1}$ by

$$\partial^Z_f(P, \alpha) = \bigcup_{n\in\mathbb{N}} \partial^Z_{f,\frac{1}{n}}(P, \alpha).$$

Fact 4. $f \in EC \iff \exists \alpha < \omega_1,\ \partial^Z_f(\mathbb{T}, \alpha) = \varnothing$.

Using this fact, we define the Zalcwasser rank as follows. For each $f \in EC$, let $|f|_Z =$ the least ordinal α for which $\partial^Z_f(\mathbb{T}, \alpha) = \varnothing$.

Fact 5. $|\cdot|_Z : EC \to \omega_1$ *is a* $\mathbf{\Pi}^1_1$*-norm.* (*See* [AK].)

Note that by a standard fact, every differentiable function has everywhere convergent Fourier series, i.e., $D(\mathbb{T}) \subseteq EC$. Ajtai and Kechris [AK] have shown that no Borel set B exists with $D(\mathbb{T}) \subseteq B \subseteq EC$. This implies the following fact:

Fact 6. *For each* $\alpha < \omega_1$, *there is a differentiable function* f *such that* $|f|_Z \geq \alpha$.

In particular, by these facts and the Boundedness Principle, $D(\mathbb{T})$ is $\mathbf{\Pi}^1_1$ non-Borel. Also the following theorem is true.

Theorem [Ajtai-Kechris]. *EC is $\mathbf{\Pi}^1_1$-complete.* (*See* [AK].)

An equivalent definition of the Zalcwasser rank

As we have seen, the definition of the Zalcwasser rank is very natural. But when we compare the Zalcwasser rank to other ranks or attempt to calculate the

Z derived set of a given closed subset and continuous function, the definition of the Zalcwasser rank is extremely difficult to work with. We give an equivalent definition of the Zalcwasser rank which is more practical. We need the following formula for Fourier series (see [Zy]).

Proposition 7. *Let δ be a fixed positive number less than π. Then*

$$S_n(f, x) - f(x) = \frac{2}{\pi}\int_0^\delta \phi_x(t)\frac{\sin nt}{t}dt + o(1), \tag{1}$$

where

$$\phi_x(t) = \frac{f(x+t) + f(x-t) - 2f(x)}{2}.$$

In this formula, $o(1)$ tends to 0 for any x and the convergence to zero is uniform in every interval where f is bounded.

Let $f \in C(\mathbb{T})$, $P \subseteq \mathbb{T}$ be a closed set, and let $x \in P$. We define $\Omega(x, f, P)$ the analogous definition of the oscillation function as follows:
(2)

$$\Omega(x, f, P) = \inf_{\delta>0}\inf_{p\in\mathbb{N}}\sup\left\{\left|\int_0^\delta \phi_y(t)\frac{\sin nt}{t}dt\right| : n \geq p \;\&\; y \in P \;\&\; |x-y| < \delta\right\}.$$

In order to calculate $\Omega(x, f, P)$, we only need to know the local behavior of f. But for $\omega(x, f, P)$, we have to calculate the nth partial sum of the Fourier series of f (which usually is not easy) before we can calculate $\omega(x, f, P)$. From this point of view, $\Omega(x, f, P)$ is more practical than $\omega(x, f, P)$. For each $f \in C(\mathbb{T})$ and each $\epsilon > 0$, we define the K derived set of P by

$$\partial^K_{f,\epsilon}(P) = \{x \in P : \Omega(x, f, P) \geq \epsilon\}.$$

As in the definition of the Zalcwasser rank, we define $\langle \partial^K_{f,\epsilon}(P, \alpha)\rangle_{\alpha<\omega_1}$ for each $\epsilon > 0$ and then $\langle \partial^K_f(P, \alpha)\rangle_{\alpha<\omega_1}$ by transfinite induction.

Theorem 8. *Let $f \in C(\mathbb{T})$ and $P \subseteq \mathbb{T}$ be a closed set. For each $\alpha < \omega_1$,*

if $f \in EC$, then $\partial^K_f(P, \alpha) = \partial^Z_f(P, \alpha)$ and if $f \notin EC$, then $\partial^K_f(P, \alpha) \neq \varnothing$.

In particular, instead of $\partial^Z_f(P, \alpha)$, we can use $\partial^K_f(P, \alpha)$ to define the Zalcwasser rank.

Proof of Theorem 8. We fix $f \in EC$. Let P be a closed subset of $\mathbb{T}$. By transfinite induction on α, it is enough to show that for each $\epsilon > 0$,

$$\partial^Z_{f,\frac{4}{\pi}\epsilon}(P, \alpha) \subseteq \partial^K_{f,\epsilon}(P, \alpha) \subseteq \partial^Z_{f,\frac{2}{\pi}\epsilon}(P, \alpha),$$

since $\partial^K_f(P, \alpha) = \bigcup_{\epsilon>0}\partial^K_{f,\epsilon}(P, \alpha)$ and $\partial^Z_f(P, \alpha) = \bigcup_{\epsilon>0}\partial^Z_{f,\epsilon}(P, \alpha)$. Hence, it suffices to show that for $x \in P$,

$$\omega(x, f, P) \geq \frac{4}{\pi}\epsilon \Rightarrow \Omega(x, f, P) \geq \epsilon \Rightarrow \omega(x, f, P) \geq \frac{2}{\pi}\epsilon.$$

Let $x \in P$. Suppose $\Omega(x, f, P) < \epsilon_1 < \epsilon_0 < \epsilon$. By the definition of $\Omega(x, f, P)$, for some $\delta > 0$, $p \in \mathbb{N}$,

$$\left|\int_0^\delta \phi_y(t)\frac{\sin nt}{t}dt\right| \leq \epsilon_1 \tag{3}$$

for all $n \geq p$ & $y \in P$ & $|x-y| < \delta$. In (3), for sufficiently large $p \in \mathbb{N}$, by (1) we have that

(4)

$$\begin{aligned}\omega(x, f, P) &\leq \sup\{|S_n(f, y) - S_m(f, y)| : n, m \geq p \;\&\; y \in P \;\&\; |x-y| < \delta\} \\ &\leq 2\sup\{|S_n(f, y) - f(y)| : n \geq p \;\&\; y \in P \;\&\; |x-y| < \delta\} \\ &\leq \frac{4}{\pi}\sup\left\{\left|\int_0^\delta \phi_y(t)\frac{\sin nt}{t}dt + o(1)\right| : n \geq p \;\&\; y \in P \;\&\; |x-y| < \delta\right\} \\ &< \frac{4}{\pi}\epsilon_0,\end{aligned}$$

since f is continuous and in (4), $o(1)$ tends to 0 uniformly on all of $\mathbb{T}$. Hence by (4), we have

$$\omega(x, f, P) \leq \frac{4}{\pi}\epsilon_0. \tag{5}$$

So $\Omega(x, f, P) < \epsilon$ implies $\omega(x, f, P) < 4\pi^{-1}\epsilon$. For the other one, suppose $\omega(x, f, P) < 2\pi^{-1}\epsilon$. Let $\epsilon_0 > 0$ such that $\omega(x, f, P) < 2\pi^{-1}\epsilon_0 < 2\pi^{-1}\epsilon$. Let $\epsilon_1 > 0$. Then for some $\delta > 0$ and $p \in \mathbb{N}$,

$$|S_n(f, y) - S_m(f, y)| \leq \frac{2}{\pi}\epsilon_0 + \epsilon_1 \tag{6}$$

for all $n, m \geq p$ & $y \in P$ & $|x-y| < \delta$. Here we can take $\delta \leq \pi$, since f is periodic. In (6) letting $m \to \infty$, we have

$$|S_n(f, y) - f(y)| \leq \frac{2}{\pi}\epsilon_0 + 2\epsilon_1 \tag{7}$$

for all $n \geq p$ & $y \in P$ & $|x-y| < \delta$. Since (1) holds uniformly in $\mathbb{T}$, by (1) and (7), we have the following:

$$\left|\frac{2}{\pi}\int_0^\delta \phi_y(t)\frac{\sin nt}{t}dt\right| \leq \frac{2}{\pi}\epsilon_0 + 2\epsilon_1,$$

for sufficiently large n and for all $y \in P$ & $|x-y| < \delta$. Hence we conclude $\Omega(x, f, P) \leq \epsilon_0 + \pi^{-1}\epsilon_1$. Since ϵ_1 is arbitrary, $\Omega(x, f, P) \leq \epsilon_0 < \epsilon$. By Fact 4 and (5), $\partial_f^K(P) \neq \varnothing$ if $f \notin EC$. It is easy to see that the second part is a consequence of the first part. □

By Fact 2, the set $b_1D(\mathbb{T})$ has arbitrary Kechris-Woodin ranks below ω_1. But for any $f \in b_1D(\mathbb{T})$, it is easy to see that the Fourier series of f converges uniformly, i.e., the Zalcwasser rank of f is 1. Hence it is natural to guess that the Kechris-Woodin rank is finer than the Zalcwasser rank. We verify this now.

Theorem 9. *For given $f \in D(\mathbb{T})$,*

$$|f|_Z \leq |f|_{KW},$$

i.e., the Kechris-Woodin rank is finer than the Zalcwasser rank.

In order to prove this, we need the following lemma.

Lemma 10. *Let $f \in D(\mathbb{T})$ and P be a closed set in $\mathbb{T}$. Then for given $\epsilon_1, \epsilon_2 > 0$,*

$$\partial^Z_{f,\epsilon_1}(P) \subseteq \partial^{KW}_{f,\epsilon_2}(P).$$

Proof of Lemma 10. Suppose $x \in P - \partial^{KW}_{f,\epsilon_2}(P)$. Then by the definition, $\exists\, \delta > 0$ such that $\forall\, p < q$, $r < s$ in $(-\delta + x, \delta + x) \cap [0, 2\pi]$ with $[p, q] \cap [r, s] \cap P \neq \varnothing$

$$|\Delta f([p, q]) - \Delta f([r, s])| < \epsilon_2. \tag{8}$$

We fix positive values δ_1, δ_2 such that $\delta_2 \leq \pi$ and $x - \delta < x - (\delta_1 + \delta_2)$. In particular, by (8),

$$2\frac{|\phi_y(t)|}{t} = |\Delta f([y - t, y]) - \Delta f([y, y + t])| \leq \epsilon_2 \tag{9}$$

holds for all $0 < t < \delta_2$ and all $y \in P \cap [x - \delta_1, x + \delta_1]$. Hence by the formula (1) and (9), we have the following: for all $y \in P \cap [x - \delta_1, x + \delta_1]$,

$$\begin{aligned}
|S_n(f, y) - f(y)| &= \left| \frac{2}{\pi} \int_0^{\delta_2} \phi_x(t) \frac{\sin nt}{t} dt \right| + o(1) \\
&\leq \frac{1}{\pi} \int_0^{\delta_2} |\Delta f([y - t, y]) - \Delta f([y, y + t])| |\sin nt| dt + o(1) \\
&\leq \frac{1}{\pi} \int_0^{\delta_2} \epsilon_2 dt + o(1) = \frac{1}{\pi} \delta_2 \epsilon_2 + o(1).
\end{aligned}$$

Since our function f is differentiable, Proposition 7 says that $o(1)$ tends to 0 uniformly in every interval, i.e., the value $o(1)$ is dependent on n only. Hence for sufficiently large n and all $y \in [x - \delta_1, x + \delta_1] \cap P$, we have

$$|S_n(f, y) - f(y)| \leq \frac{1}{\pi} \delta_2 \epsilon_2 + \delta_2,$$

i.e., $\Omega(x, f, P) \leq \pi^{-1} \delta_2 \epsilon_2 + \delta_2$ by the definition. Since δ_2 is arbitrary, we have $\Omega(x, f, P) = 0$. Hence by Theorem 8, $x \notin \partial^Z_{f,\epsilon}(P)$. So we are done. □

Proof of Theorem 9. Fix $f \in D(\mathbb{T})$. Suppose that for all ordinals $\alpha < \omega_1$, $\partial^Z_f(P, \alpha) \subseteq \partial^{KW}_f(P, \alpha)$. Since f is in $D(\mathbb{T})$, by Fact 1, there is an $\alpha < \omega_1$ such that $\partial^{KW}_f(\mathbb{T}, \alpha) = \varnothing$. Thus by our assumption, $\partial^Z_f(\mathbb{T}, \alpha)$ must be the empty set, i.e., $|f|_Z$ is less than or equal to α. Hence $|f|_Z \leq |f|_{KW}$. We show that $\partial^Z_f(P, \alpha) \subseteq \partial^{KW}_f(P, \alpha)$ by transfinite induction on α. For $\alpha = 0$ or α a limit ordinal this is obvious. Suppose it is true for α. Then by Lemma 10, we have

$$\partial^Z_f(\mathbb{T}, \alpha + 1) = \partial^Z_f(\partial^Z_f(\mathbb{T}, \alpha)) \subseteq \partial^{KW}_f(\partial^Z_f(\mathbb{T}, \alpha)).$$

It is easy to see that for all closed subsets A and B of $\mathbb{T}$ with $A \subseteq B$, $\partial^{KW}_f(A, \alpha) \subseteq \partial^{KW}_f(B, \alpha)$. Hence by the inductive assumption,

$$\partial^{KW}_f(\partial^Z_f(\mathbb{T}, \alpha)) \subseteq \partial^{KW}_f(\partial^{KW}_f(\mathbb{T}, \alpha)) = \partial^{KW}_f(\mathbb{T}, \alpha + 1).$$

Thus $\partial^Z_f(P, \alpha + 1) \subseteq \partial^{KW}_f(P, \alpha + 1)$. Hence the theorem is established. □

Construction of functions in EC having arbitrarily bad Zalcwasser rank

Recall that for each countable ordinal α, there exists a function in EC which has a Zalcwasser rank that is bigger than α. Kechris and Ajtai asked that we prove this theorem in terms of a explicit construction. Then we will have another proof that EC is non-Borel.

Theorem 11. *For each countable ordinal α, we can construct a function f in EC such that the Zalcwasser rank of f is bigger than α.*

By the Boundedness Principle, we immediately have the following corollary:

Corollary 12. *EC is $\mathbf{\Pi}^1_1$ but non-Borel.*

We introduce the Fejér polynomials. Let $N > n > 0$. We consider the polynomial

$$R(x, N, n) = -2\cos Nx \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

It is easy to show that

$$\begin{aligned} R(x, N, n) = & \frac{\sin(N-n)x}{n} + \frac{\sin(N-n+1)x}{n-1} + \cdots \\ & + \frac{\sin(N-1)x}{1} - \frac{\sin(N+1)x}{1} - \cdots - \frac{\sin(N+n)x}{n}. \end{aligned} \tag{10}$$

Since $\sum_{k=1}^{n} \sin kx/k$ is uniformly bounded in n, x, $R(x, N, n)$ is uniformly bounded in N, n, x. Let

$$\alpha_k = k^{-2}, \quad 1/2N_k = n_k = 2^{k^3} \text{ and } x_k = \frac{\pi}{4n_k}$$

for each $k \in \mathbb{N}$.

Then the series

$$\sum_{k=1}^{\infty} \alpha_k R(x, N_k, n_k) \tag{11}$$

converges to continuous functions, since $R(x, N_k, n_k)$ is uniformly bounded in x, N_k, n_k. We denote this series by $f(x)$. By unbracketing the terms of f, we can represent (11) as trigonometric series

$$\sum_{n=1}^{\infty} a_n \sin nx. \tag{12}$$

In [Zy], it is shown that (12) converges everywhere and uniformly in $[\delta, 2\pi-\delta]$ for arbitrarily small $\delta > 0$. Hence we have $\omega(x, f, \mathbb{T}) = 0$ for all $0 < x < 2\pi$. By (10), (11) and (12), we obtain

$$\begin{aligned} \left| \sum_{n=1}^{3n_k} a_n \sin nx_k - \sum_{n=1}^{2n_k} a_n \sin nx_k \right| & \geq \alpha_k \sin\frac{\pi}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_k}\right) \\ & \geq 2^{\frac{-1}{2}} \alpha_k \log n_k = 2^{\frac{-1}{2}} k \log 2 \end{aligned}$$

for each $k \in \mathbb{N}$ and each x_k. So $\omega(0, f, \{0\} \cup \{x_k\}_{k\in\mathbb{N}})$ is infinity. Let h be in EC. Recall that for any closed subset P of $\mathbb{T}$ and any $x \in P$,

$$\omega(x, f, P) \geq \frac{4}{\pi}\epsilon \Rightarrow \Omega(x, f, P) \geq \epsilon \Rightarrow \omega(x, f, P) \geq \frac{2}{\pi}\epsilon. \tag{13}$$

Hence (13) demonstates the following useful facts:

$$\begin{aligned} &\Omega(x, h, P) = 0 \iff \omega(x, h, P) = 0 \\ &\Omega(x, h, P) \text{ is positive} \iff \omega(x, h, P) \text{ is positive} \\ &\Omega(x, h, P) = \infty \iff \omega(x, h, P) = \infty. \end{aligned} \tag{14}$$

Since $\omega(0, f, \{0\} \cup \{x_k\}_{k\in\mathbb{N}}) = \infty$ and $\omega(x, f, \{0\} \cup \{x_k\}_{k\in\mathbb{N}}) = 0$ for $x \neq 0$, by (14), we get

$$\begin{aligned} &\omega(0, f, \{0\} \cup \{x_k\}_{k\in\mathbb{N}}) = \Omega(0, f, \{0\} \cup \{x_k\}_{k\in\mathbb{N}}) = \infty \text{ and} \\ &\omega(x, f, \mathbb{T}) = \Omega(x, f, \mathbb{T}) = 0, \text{ if } x \neq 0. \end{aligned} \tag{15}$$

Note that $0, 2\pi$ are the same points in $\mathbb{T}$. We are ready to prove Theorem 11.

Proof of Theorem 11. We use transfinite induction on α.

Lemma 13. *For each countable ordinal α, we can construct a function g in EC such that*

(i) $g(0) = 0$ *and* $|g|_Z = \alpha + 3$;
(ii) *for each* $\epsilon > 0$, $\partial^Z_{g,\epsilon}(\mathbb{T}, \alpha + 1) = \{0\} \cup \{x_k\}_{k\in\mathbb{N}}$;
(iii) *for each* $k \in \mathbb{N}$ *and* $\epsilon > 0$, $\Omega(x_k, g, \partial^Z_{g,\epsilon}(\mathbb{T}, \alpha)) = \infty$;
(iv) $\Omega(0, g, \{0\} \cup \{x_k\}_{k\in\mathbb{N}}) = \infty$.

Proof of Lemma 13. Let a, b, c be real numbers. Let P be a closed subset of $\mathbb{T}$. For a continuous function h on $\mathbb{T}$, we define $h_{a,b,c}$, $P_{b,c}$ and $\phi_x(h, t)$ as follows:
$h_{a,b,c}(x) = ah(b(x - c))$ and $P_{b,c} = \{c + x/b : x \in P\}$;
$\phi_x(h, t) = (h(t + x) + h(-t + x) - 2h(x))/2$.

We will require the following lemma:

Lemma 14. *Let h be in EC and $x \in \mathbb{T}$. Let P be a closed subset of $\mathbb{T}$. Suppose $\Omega(x, h, P) = \infty$. Then for any $a \neq 0$, $N \in \mathbb{N}$ and $b \in \mathbb{T}$, the value of the oscillation function of $h_{a,N,b}$ at $b + x/N$ is infinity, i.e., $\Omega(b + x/N, h_{a,N,b}, P_{N,b}) = \infty$.*

Lemma 14 has the most important role to play in our construction. Namely, if the value of oscillation function of a function at a point is infinity, so is the value of the oscillation function of a function at the scaled point after scaling an interval. In order to prove Lemma 14, one should focus on local behaviors of functions. The new defintion of Zalcwasser rank in Theorem 8, i.e., the oscillation function, is based on local propeties of functions. So we are able to prove Lemma 14.

Proof of Lemma 14. Let $\delta > 0$ and $p \in \mathbb{N}$. We then obtain

$$\begin{aligned}
&\sup\left\{\left|\int_0^\delta \phi_y(h_{a,N,b}, t)\frac{\sin mt}{t}dt\right| : m \geq p,\, y \in P_{N,b} \text{ and } |y-b-x/N| < \delta\right\}\\
&= |a|\sup\left\{\left|\int_0^\delta \phi_{N(y-b)}(h, Nt)\frac{\sin mt}{t}dt\right| : m \geq p,\, N(y-b) \in P\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\text{and } |N(y-b)-x| < N\delta\right\}\\
&= |a|\sup\left\{\left|\int_0^\delta \phi_z(h, Nt)\frac{\sin mt}{t}dt\right| : m \geq p,\, z \in P \text{ and } |z-x| < N\delta\right\}\\
&= |a|\sup\left\{\left|\int_0^{N\delta} \phi_z(h, t)\frac{\sin mt/N}{t}dt\right| : m \geq p,\, z \in P \text{ and } |z-x| < N\delta\right\}\\
&\geq |a|\sup\left\{\left|\int_0^{N\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| : n \geq \frac{p}{N},\, z \in P \text{ and } |z-x| < \delta\right\}.
\end{aligned}$$

By (2), we get

$$\begin{aligned}
&\Omega(b+x/N, h_{a,N,b}, P_{N,b})\\
&\quad\geq \inf_{\delta>0}\inf_{p\in\mathbb{N}}|a|\sup\left\{\left|\int_0^{N\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| : n \geq \frac{p}{N},\, z \in P \text{ and } |z-x| < \delta\right\}.
\end{aligned}$$

So in order to prove our lemma, it is enough to show that for each $\delta > 0$ and $p \in \mathbb{N}$,

$$\sup\left\{\left|\int_0^{N\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| : n \geq \frac{p}{N},\, z \in P \text{ and } |z-x| < \delta\right\} = \infty.$$

We let $M = \max\{|\phi_z(h, t)| : z, t \in \mathbb{T}\}$. Clearly, $0 \leq M < \infty$, since h is continuous. For each $n \geq p/N$ and $z \in P$ with $|z-x| < \delta$, we have

(16)
$$\begin{aligned}
\left|\int_0^{N\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| &\geq \left|\int_0^{\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| - \left|\int_\delta^{N\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right|\\
&\geq \left|\int_0^{\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| - \int_\delta^{N\delta} |\phi_z(h, t)|\frac{|\sin nt|}{t}dt
\end{aligned}$$

By the definition of M and (16), we obtain the follwoing:

$$\begin{aligned}
\left|\int_0^{N\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| &\geq \left|\int_0^{\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| - M(N-1)\delta\frac{1}{\delta}\\
&= \left|\int_0^{\delta} \phi_z(h, t)\frac{\sin nt}{t}dt\right| - M(N-1).
\end{aligned}$$

Hence we get the following:

$$\sup\left\{\left|\int_0^{N\delta}\phi_z(h,t)\frac{\sin nt}{t}dt\right| : n\geq\frac{p}{N},\ z\in P \text{ and } |z-x|<\delta\right\}$$
$$\geq \sup\left\{\left|\int_0^{\delta}\phi_z(h,t)\frac{\sin nt}{t}dt\right| : n\geq\frac{p}{N},\ z\in P \text{ and } |z-x|<\delta\right\}-M(N-1)$$
$$=\infty,$$

for each $\delta>0$ and $p\in\mathbb{N}$. We conclude $\Omega(b+x/N, h_{a,N,b}, P_{N,b})=\infty$. The proof of Lemma 14 follows. □

We divide the proof into three cases.

Case 1. $\alpha=0$.

We choose a sequence of positive integers $\langle M_k\rangle_{k\in\mathbb{N}}$ satisfying the following: for each $k\in\mathbb{N}$,

(1) $x_{k+1}+\pi/M_{k+1}<x_k-1/M_k$ and $8n_k$ divides M_k.

We define $I_k=\left(x_k-\pi/M_k, x_k+\pi/M_k\right)$ for each $k\in\mathbb{N}$. We take a sequence of positive real numbers $\langle a_k\rangle_{k\in\mathbb{N}}$ converging to 0. Let

$$g_1(x)=\begin{cases} a_k f\left(M_k(x-x_k)\right), & \text{if } x\in I_k;\\ 0, & \text{otherwise.}\end{cases}$$

Note that for each $k\in\mathbb{N}$, $f\left(M_k(x_k\pm\pi/M_k-x_k)\right)=0$. Hence g_1 is continuous except for $x=0$. Clearly, $g_1(0)=0$. Since a_k converges to 0, it is easy to see that g_1 is also continuous at $x=0$. By (15) and Lemma 14, we have

$$\Omega(x_k, g_1, \mathbb{T})=\infty,$$

for all $k\in\mathbb{N}$. If $x\neq 0, x_k$, then we get $\omega(x, g_1, \mathbb{T})=0$. By (14), so is $\Omega(x, g_1, \mathbb{T})$. Hence g_1 satisfies (ii) and (iii). Suppose

$$\Omega\left(0, g_1, \{0\}\cup\{x_k\}_{k\in\mathbb{N}}\right)=\infty.$$

Then let $g=g_1$. The continuous function g fulfills $g(0)=0$, (ii), (iii) and (iv). Clearly, (ii), (ii) and (iii) imply $|g|=3$. By Fact 4, f is in EC. So we are done. Suppose otherwise. We let $g=g_1+f$. Still, g fulfills (ii) and (iii). For (iv), by (2) and (15), it is enough to show that for any $\delta>0$ and $p\in\mathbb{N}$,

$$\sup\left\{\left|\int_0^{\delta}\phi_x(g_1,t)\frac{\sin nt}{t}dt\right| : n\geq p \text{ and } x=x_k<\delta\right\}<\infty. \tag{17}$$

Suppose (17) does not hold for some $\delta>0$ and $p\in\mathbb{N}$, i.e.,

$$\sup\left\{\left|\int_0^{\delta}\phi_x(g_1,t)\frac{\sin nt}{t}dt\right| : n\geq p \text{ and } x=x_k<\delta\right\}=\infty.$$

Let $\delta_1>0$ and $p_1\in\mathbb{N}$. Let $A=\max\{|g_1(t)| : t\in\mathbb{T}\}$. Fix $m\in\mathbb{N}$. For any $y\in\mathbb{T}$, it is well known that $\int_0^\delta\phi_y(g_1,t)\sin nt/t\,dt$ converges to 0, i.e.,

$$0\leq\sup\left\{\left|\int_0^{\delta}\phi_y(g_1,t)\frac{\sin nt}{t}dt\right| : n\in\mathbb{N}\right\}<\infty. \tag{18}$$

For any $q \in \mathbb{N}$, it is easy to see that

$$0 \leq \sup \left\{ \left| \int_0^{\delta} \phi_z(g_1, t) \frac{\sin qt}{t} dt \right| : z \in \mathbb{T} \right\} < \infty. \tag{19}$$

Hence by (18) and (19), we may find $k = k(m)$, $n = n(m)$ such that $x = x_k < \delta_1$, $n(m) \geq p_1$ and

$$\left| \int_0^{\delta} \phi_x(g_1, t) \frac{\sin nt}{t} dt \right| > m. \tag{20}$$

From (20), we have

$$\begin{aligned} \left| \int_0^{\delta_1} \phi_x(g_1, t) \frac{\sin nt}{t} dt \right| &\geq \left| \int_0^{\delta} \phi_x(g_1, t) \frac{\sin nt}{t} dt \right| - \left| \int_{\delta}^{\delta_1} 4M \frac{1}{t} dt \right| \\ &> m - 4|\log(\delta_1/\delta)|. \end{aligned}$$

Since m is arbitrary, we have

$$\sup \left\{ \left| \int_0^{\delta_1} \phi_x(g_1, t) \frac{\sin nt}{t} dt \right| : n \geq p_1 \text{ and } x = x_k < \delta_1 \right\} = \infty,$$

for all $\delta_1 > 0$ and $p_1 \in \mathbb{N}$. Hence we derive $\Omega(0, g_1, \{0\} \cup \{x_k\}_{k \in \mathbb{N}}) = \infty$. It contradicts our assumption. g satisfies (ii), (iii) and (iv). As before, g fulfills all conditions of the lemma. We have shown case 1.

Case 2. α is a successor ordinal, i.e., $\alpha = \beta + 1$ for some $\beta < \omega_1$.

The proof of case 2 is similar to case 1. By induction, we get a function g_β in EC fulfilling all conditions of our lemma. We take sequences $\langle M_k \rangle_{k \in \mathbb{N}}$ and $\langle a_k \rangle_{k \in \mathbb{N}}$ as in case 1. We let

$$g_1(x) = \begin{cases} a_k g_\beta(M_k(x - x_k)), & \text{if } x \in I_k; \\ 0, & \text{otherwise.} \end{cases}$$

By Fact 4, g_1 is a function in EC. By g_1, (14), (15), Lemma 14 and the same method as in case 1, we are able to construct a function g in EC that satisfies (i), (ii), (iii) and (iv). Note that (iv) has a role to play in this construction.

Case 3. α is a limit ordinal.

As in case 1, it is enough to show that we construct a continuous function satisfying only (i), (ii) and (iii). We choose a sequence of ordinals $\langle \beta_n \rangle_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\beta_n < \alpha$ and $\sup_{n \to \infty} \beta_n = \alpha$. By transfinite induction, we get a sequence of continuous functions $\langle g_{\beta_n} \rangle_{n \in \mathbb{N}}$ satisfying (i), (ii), (ii) and (iv). Let $\langle a_k \rangle$ and $\langle M_k \rangle$ as in case 1. Clearly, the function

$$g_1(x) = \begin{cases} a_k g\beta_n(M_k(x - x_k)), & \text{if } x \in I_k; \\ 0, & \text{otherwise.} \end{cases}$$

satisfies $\partial^Z_{g_1, \epsilon}(\mathbb{T}, \alpha) = \{0\}$ for each $\epsilon > 0$. By the same way, we can construct a continuous function g_2 such that $\partial^Z_{g_2, \epsilon}(\mathbb{T}, \alpha + 1) = \{0\} \cup \{x_k\}_{k \in \mathbb{N}}$. Note that the condition (iii) and (iv) have important roles to play in this construction. Without loss of generality, g_2 has (iii), since for each $k \in \mathbb{N}$, $\epsilon > 0$, if

$\Omega(x_k, g_2, \partial^Z_{g_2,\epsilon}(\mathbb{T}, \alpha)) < \infty$, then g_3 has the same properties as g_2 except for $\Omega(x_k, g_3, \partial^Z_{g_2,\epsilon}(\mathbb{T}, \alpha)) = \infty$, where

$$g_3(x) = \begin{cases} g_2(x) + a_k f(M_k(x - x_k)), & \text{if } x \in I_k; \\ g_2(x), & \text{otherwise.} \end{cases}$$

As in case 1, we finally can construct a continuous function g satisfying all conditions of Lemma 13.

By case 1, case 2 and case 3, we complete the proof of Lemma 13.

By Theorem 8 and Lemma 13, Theorem 11 follows. □

The Denjoy rank and a remark

For each $f \in D(\mathbb{T})$, there is the canonical rank, which is called the Denjoy rank, $|f|_{DJ}$, from $D(\mathbb{T})$ to ω_1 and it measures how long it takes to recover f from f' via the Denjoy process (see [Br]). We quickly introduce the Denjoy rank. Let g be a measurable function on $\mathbb{T}$ and P be a closed subset of $\mathbb{T}$. We define the set of all singular points of g over P by

$$S(g, P) = \{x \in P : g \text{ is not Lebesgue integrable on } I \cap P \text{ for any open interval } I \text{ with } x \in I\}.$$

Let $f \in C(\mathbb{T})$. Let $\langle (a_n, b_n) \rangle$ be the sequence of open intervals complementing P in $\mathbb{T}$. We define the set of divergence points of f over P by

$$D(f, P) = \{x \in P : \sum_I |f(b_n) - f(a_n)| \text{ diverges for every open interval } I \text{ with } x \in I\}.$$

Here $\sum_I$ indicates that the sum is to be taken over all the intervals (a_n, b_n) which are contained in I. For $f \in D(\mathbb{T})$ and each closed subset P of $\mathbb{T}$, we define the DJ derived set of P by

$$\partial^{DJ}_f(P) = S(f', P) \cup D(f, P).$$

As before, we define the transfinite sequence $\langle \partial^{DJ}_f(P, \alpha) \rangle_{\alpha < \omega_1}$. For each $f \in D(\mathbb{T})$, let $|f|_{DJ} =$ the least ordinal α for which $\partial^{DJ}_f(\mathbb{T}, \alpha) = \varnothing$. For $f \in D(\mathbb{T})$, it is known that $|f|_{DJ} = 1$ if and only if f' is integrable. Hence $|f|_{DJ} = 1$ implies that the Fourier series of f converges uniformly, i.e., $|f|_Z = 1$. So we might guess that $|f|_Z \leq |f|_{DJ}$.

Conjecture. For each $f \in D(\mathbb{T})$, $|f|_Z \leq |f|_{DJ}$.

T. Ramsamujh [Ra] has shown that $|f|_{DJ} \leq |f|_{KW}$. In fact he proved that for any $\epsilon > 0$ and $\alpha < \omega_1$, $\partial^{DJ}_f(P, \alpha) \subseteq \partial^{KW}_{f,\epsilon}(P, \alpha)$. We have shown that for any $\epsilon_1, \epsilon_2 > 0$ and $\alpha < \omega_1$, $\partial^Z_{f,\epsilon_1}(P, \alpha) \subseteq \partial^{KW}_{f,\epsilon_2}(P, \alpha)$. So it seems likely that $\partial^Z_{f,\epsilon}(P, \alpha) \subseteq \partial^{DJ}_f(P, \alpha)$ for any $\epsilon > 0$ and α, i.e., $|f|_Z \leq |f|_{DJ}$. Hence our result can be viewed as evidence that this conjecture is true. If indeed the conjecture holds, our theorem would be an immediate corollary to it and Ramsamujh's result, and we would have that $|f|_Z \leq |f|_{DJ} \leq |f|_{KW}$.

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Note added in proof. Recently, we have shown that this conjecture is not true. Namely for any ordinal α, any nonzero ordinal β and any countable ordinal γ with $\alpha, \beta < \gamma$, there exists a differentiable function f on the unit circle such that

$$|f|_Z = \alpha + 1, \quad |f|_{DJ} = \beta + 1 \quad \text{and} \quad |f|_{KW} = \gamma.$$

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